

## TMD Evolution at Moderate Hard Scales

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We summarize some of our recent work on non-perturbative transverse momentum dependent (TMD) evolution, emphasizing aspects that are necessary for dealing with moderately low scale processes like semi-inclusive deep inelastic scattering.

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## 1. TMD factorization and non-perturbative evolution

The purpose of this talk is to summarize results recently presented in Ref. [1]. We will discuss the Collins-Soper-Sterman (CSS) form of TMD factorization in the updated version presented in Ref. [2]. (See Ref. [3] for a general overview and for references.) For these proceedings, the relevant aspects of the TMD factorization theorems are the following:

- The unpolarized cross section for a process like Drell-Yan scattering is expressible as

$$\begin{aligned} & \frac{d\sigma}{d^4q d\Omega} \\ &= \frac{2}{s} \sum_j \frac{d\hat{\sigma}_{j\bar{j}}(Q, \mu \rightarrow Q; \alpha_s(Q))}{d\Omega} \int d^2\mathbf{b} e^{iq_T \cdot \mathbf{b}} \tilde{F}_{j/A}(x_A, \mathbf{b}; Q^2, Q) \tilde{F}_{\bar{j}/B}(x_B, \mathbf{b}; Q^2, Q) \\ &+ \text{large } q_T \text{ “Y-term” correction.} \end{aligned} \quad (1.1)$$

where  $d\hat{\sigma}_{j\bar{j}}/d\Omega$  is a hard partonic cross section and  $\tilde{F}(x, \mathbf{b}; Q^2, Q)$  is a TMD parton distribution function (PDFs) in coordinate space evaluated with a hard scale  $Q$ .

- Collins-Soper (CS) evolution applied to an individual TMD PDF leads to

$$\frac{\partial}{\partial \ln Q} \ln \tilde{F}_{j/A}(x_A, \mathbf{b}_T; Q^2, Q) = \tilde{K}(b_T; Q) + b_T \text{ Independent Terms} \quad (1.2)$$

The “ $b_T$  Independent Terms” only affect the *normalization* of  $\tilde{F}$  but not its shape.

- The kernel  $\tilde{K}(b_T; Q)$  is strongly universal. At small  $b_T$  its  $b_T$ -dependence is perturbatively calculable with  $1/b_T$  acting as a hard scale. At large  $b_T$  its  $b_T$ -dependence is non-perturbative.
- For all  $b_T$ ,  $\tilde{K}(b_T; Q)$  obeys the renormalization group (RG) equation:

$$\frac{d}{d \ln \mu} \tilde{K}(b_T; \mu) = -\gamma_K(\alpha_s(\mu)). \quad (1.3)$$

At small  $b_T$ , one hopes to exploit perturbation theory with  $1/b_T$  as a hard scale to calculate  $\tilde{K}(b_T; Q)$  while at large  $b_T$  a non-perturbative parametrization is needed. In the non-perturbative region, one hopes to exploit the strong universality of  $\tilde{K}(b_T; Q)$  to make predictions. One needs a prescription to demarcate what constitutes large and small  $b_T$ . To smoothly interpolate between the two regions, one imposes a gentle cutoff on large  $b_T$ . A common choice of cutoff function is

$$\mathbf{b}_*(\mathbf{b}_T) = \frac{\mathbf{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}. \quad (1.4)$$

Then an RG scale defined as  $\mu_{b_*} \equiv C_1/b_*$  approaches  $C_1/b_T$  at small  $b_T$  and  $C_1/b_{\max}$  at large  $b_T$ . We can separate  $\tilde{K}(b_T; Q)$  into a large  $b_T$  part and a small  $b_T$  part by adding and subtracting  $\tilde{K}(b_*; Q)$  in Eq. (1.2):

$$\frac{\partial}{\partial \ln Q} \ln \tilde{F}_{j/A}(x_A, \mathbf{b}_T; Q^2, Q) = \tilde{K}(b_*; Q) + [\tilde{K}(b_T; Q) - \tilde{K}(b_*; Q)] + b_T \text{ Independent Terms}. \quad (1.5)$$

The  $g_K(b_T; b_{\max})$  function is defined as the term  $\tilde{K}(b_T; Q) - \tilde{K}(b_*; Q)$ , so that

$$\frac{\partial}{\partial \ln Q} \ln \tilde{F}_{j/A}(x_A, \mathbf{b}_T; Q^2, Q) = \tilde{K}(b_*; Q) - g_K(b_T; b_{\max}) + b_T \text{ Independent Terms}. \quad (1.6)$$

By definition, the right side of Eq. (1.6) is exactly independent of  $b_{\max}$ . From Eq. (1.3),  $g_K(b_T; b_{\max})$  is also exactly independent of  $Q$ . The  $Q$  dependence in each of the terms in the definition of  $g_K(b_T; b_{\max})$  cancels. We can apply Eq. (1.3) to exploit RG improvement in the calculation of  $\tilde{K}(b_*; Q)$ :

$$\tilde{K}(b_*; Q) = \tilde{K}(b_*; \mu_{b_*}) - \int_{\mu_{b_*}}^Q \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')). \quad (1.7)$$

So, the evolution of the *shape* of  $\tilde{F}_{j/A}(x_A, \mathbf{b}_T; Q^2, Q)$  is given by

$$\begin{aligned} & \frac{\partial}{\partial \ln Q} \ln \tilde{F}_{j/A}(x_A, \mathbf{b}_T; Q^2, Q) \\ &= \tilde{K}(b_*; \mu_{b_*}) - \int_{\mu_{b_*}}^Q \frac{d\mu'}{\mu'} \gamma_K(\alpha_s(\mu')) - g_K(b_T; b_{\max}) + b_T \text{ Independent Terms}. \end{aligned} \quad (1.8)$$

The partial derivative symbol means  $x_A$  is to be held fixed. The  $g_K(b_T; b_{\max})$  function inherits the universality properties of  $\tilde{K}(b_T; \mu)$ . In particular, it is related to the *vacuum* expectation value of a relatively simple Wilson loop. It is independent of any details of the process and is even the same if the PDF  $\tilde{F}_{j/A}(x_A, \mathbf{b}_T; Q^2, Q)$  is replaced with a fragmentation function. Thus we say that  $g_K(b_T; b_{\max})$  is “strongly” universal; see the graphic in Fig. 1. The  $g_K(b_T; b_{\max})$  function is often called the “non-perturbative” part of the evolution since it can contain non-perturbative elements. This is a slight misnomer, however, since  $g_K(b_T; b_{\max})$  can contain perturbative contributions as well. Indeed, at very small  $b_T$  it is entirely perturbatively calculable, though suppressed by powers of  $b_T/b_{\max}$ , according to its definition in Eq. (1.5).

## 2. Large $b_T$ behavior

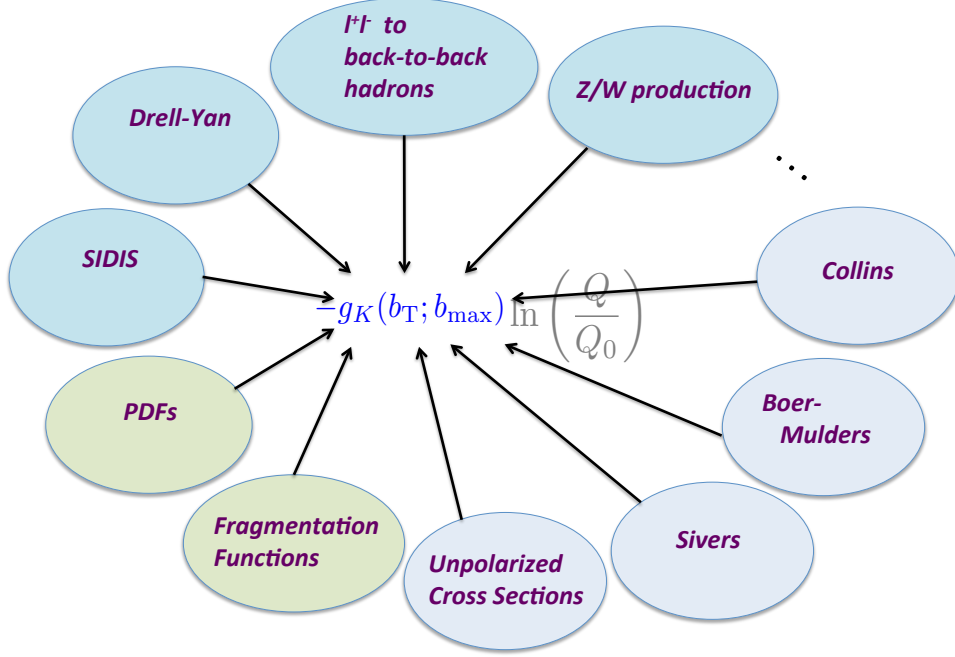
A common choice for non-perturbative parametrizations of  $g_K(b_T; b_{\max})$  is a power-law form. These tend to yield reasonable success in fits that involve at least moderately high scales  $Q$  [4]. However, extrapolations of those fits to lower values of  $Q$  (such as those corresponding to many current SIDIS experiments) appear to produce evolution that is far too rapid [5, 6]. In this talk, we carefully examine the underlying physics issues surrounding non-perturbative evolution and, on the basis of those considerations, we will propose a form for  $g_K(b_T; b_{\max})$  that accommodates both large and small  $Q$  behavior.

We will first write down our proposed ansatz for  $g_K(b_T; b_{\max})$  and then spend the remainder of the talk discussing its justifications. Our proposal is

$$g_K(b_T; b_{\max}) = g_0(b_{\max}) \left( 1 - \exp \left[ - \frac{C_F \alpha_s(\mu_{b_*}) b_T^2}{\pi g_0(b_{\max}) b_{\max}^2} \right] \right), \quad (2.1)$$

where

$$g_0(b_{\max}) = g_0(b_{\max,0}) + \frac{2C_F}{\pi} \int_{C_1/b_{\max,0}}^{C_1/b_{\max}} \frac{d\mu'}{\mu'} \alpha_s(\mu'). \quad (2.2)$$



**Figure 1:** Strong universality of the non-perturbative evolution parametrized by  $g_K(b_T; b_{\max})$ . The  $-g_K(b_T; b_{\max}) \ln(Q/Q_0)$  combination appears exponentiated in the evolved cross section expression.

The only parameter of the model is  $g_0(b_{\max})$  and it varies with  $b_{\max}$  according to Eq. (2.2).  $b_{\max,0}$  is a boundary value for  $g_0$  relative to which other values are determined.

First, note that the a small  $b_T/b_{\max}$  expansion of Eq. (2.1) gives

$$g_K(b_T; b_{\max}) = \frac{C_F}{\pi} \frac{b_T^2}{b_{\max}^2} \alpha_s(\mu_{b_*}) + O\left(\frac{b_T^4 C_F^2 \alpha_s(\mu_{b_*})^2}{b_{\max}^4 \pi^2 g_0(b_{\max})}\right), \quad (2.3)$$

while an expansion of the exact definition of  $-g_K(b_T; b_{\max})$  in Eq. (1.5) is

$$\begin{aligned} g_K(b_T; b_{\max}) &= -\tilde{K}(b_T; \mu_{b_*}; \alpha_s(\mu_{b_*})) + \tilde{K}(b_*; \mu_{b_*}; \alpha_s(\mu_{b_*})) \\ &= \frac{C_F}{\pi} \frac{b_T^2}{b_{\max}^2} \alpha_s(\mu_{b_*}) + O\left(\frac{b_T^4}{\pi^2 b_{\max}^4} \alpha_s(\mu_{b_*})^2\right) \end{aligned} \quad (2.4)$$

So, the exact definition and Eq. (2.1) match in the small  $b_T$  limit.

### 3. Conditions on $g_K(b_T; b_{\max})$

Our description of the large  $b_T$  limit of correlation functions like  $\tilde{F}(x_A, \mathbf{b}_T; Q^2, Q)$  is motivated by the general observation that the analytic properties of correlation functions imply an exponential

coordinate dependence, with a possible power-law fall-off, for the large  $b_T$  limit. That is, neglecting perturbative contributions,

$$\tilde{F}(x_A, \mathbf{b}_T; Q^2, Q) \stackrel{b_T \rightarrow \infty}{\sim} \frac{1}{b_T^\alpha} e^{-mb_T}, \quad (3.1)$$

with  $m$  and  $\alpha$  independent of  $Q$ . See, for example, Ref. [7]. Therefore, from Eq. (1.2),  $\tilde{K}(b_T; Q)$  must approach a  $b_T$ -independent constant at large  $b_T$ .

The set of requirements on  $g_K(b_T; b_{\max})$  is

1.  $\tilde{K}(b_T; \mu_{b_*}) \stackrel{b_T \rightarrow 0}{=} \tilde{K}(b_T; C_1/b_T)$  is calculable entirely in perturbation theory with  $C_1/b_T$  playing the role of a hard scale.
2.  $\tilde{K}(b_T; Q)$  approaches a constant at  $b_T/b_{\max} \rightarrow \infty$ . The constant can be  $Q$ -dependent, but the  $Q$ -dependence can be calculated perturbatively for all  $b_T$  from Eq. (1.3).
3. Because of item 2,  $g_K(b_T; b_{\max})$  must approach a constant at large  $b_T$ , but the constant depends on  $b_{\max}$ .
4. At small  $b_T$ ,  $g_K(b_T; b_{\max})$  is a power series in  $(b_T/b_{\max})^2$  with perturbatively calculable coefficients, as in Eqs. (2.3,2.4).
5. By definition, the right side of Eq. (1.8) is independent of  $b_{\max}$  and this should be preserved as much as possible in the functional form that parametrizes  $g_K(b_T; b_{\max})$ . For small  $b_T$ , this means

$$\left. \text{asy}_{b_T \ll b_{\max}} \frac{d}{db_{\max}} g_K(b_T; b_{\max}) \right|_{\text{parametrized}} = \left. \text{asy}_{b_T \ll b_{\max}} \frac{d}{db_{\max}} g_K(b_T; b_{\max}) \right|_{\text{truncated PT}} \quad (3.2)$$

where “parametrized” refers to a specific model of  $g_K(b_T; b_{\max})$  while “truncated PT” refers to a truncated perturbative expansion. Eqs. (2.3,2.4) satisfy this requirement through order  $\alpha_s(\mu_{b_*})$ .

6. At large  $b_T$ ,  $b_{\max}$ -independence of the exact  $\tilde{K}(b_T, \mu)$  implies that, to a useful approximation,

$$\frac{d}{d \ln b_{\max}} g_K(b_T = \infty; b_{\max}) = \left[ \frac{d \tilde{K}(b_{\max}; C_1/b_{\max})}{d \ln b_{\max}} - \gamma_K(\alpha_s(C_1/b_{\max})) \right]_{\text{truncated PT}}, \quad (3.3)$$

as obtained from Eq. (1.7) and the definition of  $g_K$ . Equation (2.2) ensures that Eq. (2.1) satisfies Eq. (1.8) so long as everything is calculated only to order  $\alpha_s(\mu_{b_*})$ . Enforcing both Eq. (3.2) and Eq. (3.3) simultaneously means  $g_K(b_T; b_{\max})$  will produce a  $b_{\max}$  independent contribution to  $\tilde{K}(b_T; Q)$  for all  $b_T$  except perhaps for an intermediate region at the border between perturbative and non-perturbative  $b_T$ -dependence. The residual  $b_{\max}$  dependence there can be reduced by calculating higher orders and refining knowledge of non-perturbative behavior.

For a much more detailed discussion of these considerations, see Sect. VII of Ref. [1]. Equation (2.1) is one of the simplest models that satisfies all 6 of these properties simultaneously.

#### 4. Conclusion

In Sect. 3 we enumerated properties that a model of  $g_K(b_T; b_{\max})$  needs to ensure basic consistency in a calculation  $\tilde{K}(b_T; Q)$ . A simple parametrization was proposed in Sect. 2.

Note that a quadratic  $(b_T/b_{\max})^2$  dependence at small  $b_T$  emerges naturally from (2.1), but with a perturbatively calculable coefficient. Furthermore, the dependence is not exactly quadratic because the coefficients contain logarithmic  $b_T$  dependence through  $\alpha_s(\mu_{b_*})$ .

In a process dominated by very large  $b_T$ , Sect. 3 and Eq. (2.1) predict an especially simple evolution for the low- $Q$  cross section. Namely, the cross section scales as  $(Q/Q_0)^a$  where  $a$  is combination of  $g_K(\infty, b_{\max})$  and perturbatively calculable quantities. (See Eq. (85,86) of Ref. [1].)

Future phenomenological work should include efforts to constrain  $g_0$ . Because of its strongly universal nature, this offers a relatively simple way to test TMD factorization.

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